Non-perturbative regularization and renormalization: simple examples from non-relativistic quantum mechanics

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Abstract

We examine several zero-range potentials in non-relativistic quantum mechanics. The study of such potentials requires regularization and renormalization. We contrast physical results obtained using dimensional regularization and cutoff schemes and show explicitly that in certain cases dimensional regularization fails to reproduce the results obtained using cutoff regularization. First we consider a delta-function potential in arbitrary space dimensions. Using cutoff regularization we show that for $d \geq 4$ the renormalized scattering amplitude is trivial. In contrast, dimensional regularization can yield a nontrivial scattering amplitude for odd dimensions greater than or equal to five. We also consider a potential consisting of a delta function plus the derivative-squared of a delta function in three dimensions. We show that the renormalized scattering amplitudes obtained using the two regularization schemes are different. Moreover we find that in the cutoff-regulated calculation the effective range is necessarily negative in the limit that the cutoff is taken to infinity. In contrast, in dimensional regularization the effective range is unconstrained. We discuss how these discrepancies arise from the dimensional regularization prescription that all power-law divergences vanish. We argue that these results demonstrate that dimensional regularization can fail in a non-perturbative setting.

1 Introduction

The technique known as dimensional regularization (DR) [1] is the method of choice for dealing with the infinities which appear in perturbative quantum field theory. This elegant approach preserves symmetries, e.g. gauge invariance and chiral symmetry, while eliminating power-law divergences and isolating all logarithmic divergences. It is natural to ask why the neglect of power-law divergences in DR is justified. In fact, the Bogoliubov-Parasuik-Hepp-Zimmermann (BPHZ) renormalization scheme, in which subtractions are applied directly to the integrand in divergent expressions for loop graphs, allows perturbative renormalization to proceed without the specification of *any* regularization scheme [2]. Thus, given Wilson's proof that DR is a unique procedure which is consistent with ordinary integration for finite integrals [3] it follows that, after renormalization, perturbative calculations using DR yield results identical to those obtained using other forms of regularization. However, it has *not* been shown that DR is equivalent to these other forms of regularization in non-perturbative calculations.

In this paper, we discuss some examples involving the non-perturbative regularization and renormalization of delta-function potentials in non-relativistic quantum mechanics [4, 5, 6]. In some of these examples DR gives physical results which differ from those obtained using Pauli-Villars or cutoff regularization. Our examples explicitly illustrate the way in

which DR can fail in a non-perturbative context. They are also potentially of practical significance, since delta-function potentials of the type discussed here appear in effective field theory attempts to describe the nucleon-nucleon interaction for momenta well below the pion mass [7, 8, 9]. Indeed, the issue of whether a series of delta functions and their derivatives can systematically model a more fundamental potential at low momenta is inextricably linked with regularization, and will be discussed elsewhere [10]. The potentials discussed in this paper will be treated as exact and not as the leading terms in an expansion.

Our examples involve solving the Schrödinger (or Lippmann-Schwinger) equation for the S-wave scattering of a spinless particle off a potential which is the sum of delta functions and derivatives of delta functions. Here we consider potentials of the form

$$\langle x|V_d^{(n)}|x'\rangle = [C + C_2(\nabla^2 + \nabla'^2) + C_4(\nabla^4 + \nabla'^4) + C_{22}\nabla^2\nabla'^2 + \ldots]\delta^{(d)}(x - x')\delta^{(d)}(x), \quad (1)$$

where x and x' are d-dimensional Euclidean vectors. The potential $V_d^{(n)}$ includes up to n derivatives of the delta function.

It is important to recognize that if the coefficients $C, C_2, \ldots$ are finite then the Hamiltonian H, defined by

$$H = \frac{\hat{p}^2}{2\mu} + V_d^{(n)}, \quad (2)$$

where μ is the mass of the particle, is not meaningful in any integer dimension $d > 1$. One may see this by studying the Born series for $V_d^{(0)}$. The Nth term in this series can be represented as an $N-1$-loop diagram. (See Fig. 1.) Solving the Lippmann-Schwinger equation involves summing this entire series. If $d > 1$ then for any finite value of the coefficient C the graphs shown in Fig. 1 are divergent for all $N \geq 2$. So, in general, only after regularization and renormalization will $V_d^{(n)}$ yield finite results.

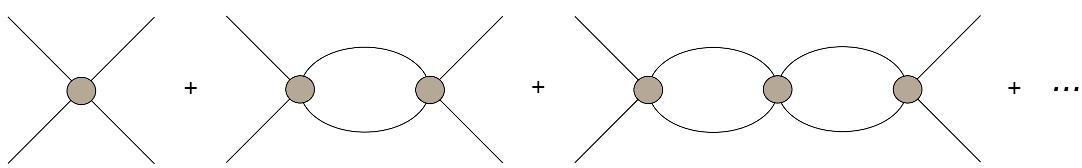

Figure 1: The Born series for $V_d^{(0)}$, which is represented by the shaded blob.

One way to regularize and renormalize is as follows. First, all divergent integrals are regulated by the introduction of a sharp or smooth cutoff, thereby making the otherwise-divergent Born series finite. This now-finite scattering amplitude is then renormalized by choosing the coefficients to be appropriate functions of the cutoff. These functions are chosen in such a way that certain physical observables are reproduced. Finally, by taking the limit of the resulting amplitude when the cutoff goes to infinity, one recovers results corresponding to the original potential. In other words, one obtains finite results by choosing the coefficients $C, C_2, \ldots$ to go to zero in a specific way. All cutoff schemes yield identical results for the renormalized amplitude derived from $V_d^{(n)}$, provided that the cutoff is taken to infinity.

Such schemes represent an obvious prescription for obtaining a finite scattering amplitude from the potential (1). As noted above, solving the Schrödinger equation is intrinsically non-perturbative, involving the iteration of the potential to all orders. If such a calculation were done order-by-order in the loop expansion then, since this is a non-renormalizable theory, an infinite number of counterterms would have to be introduced into the amplitude. On the other hand, results may be obtained upon the introduction of a finite number of counterterms in the potential. However, if such non-perturbative renormalization is used the equivalence of different regularization schemes cannot be taken for granted. Thus, a natural question is whether using DR to render the integrals in the Born series finite would, once renormalization is performed, lead to the same scattering amplitude.

In Sec. 2 we discuss this question for $V_d^{(0)}$, a d-dimensional delta function. A wide variety of regularization schemes have been used to give this potential meaning [4, 5, 6]. Moreover, Friedman has shown that for any $d > 1$ repulsive delta-function potentials are trivial, while for any $d > 3$ all delta-function potentials are trivial, i.e. the scattering amplitude goes to zero when the regulator is removed from the problem [11, 12]. To our knowledge, results obtained when DR is used to regulate the delta-function potential do not exist in the literature for any $d \neq 3$. We find that the renormalized amplitudes obtained using the DR and cutoff schemes disagree in any odd dimension $d \geq 5$. There the renormalized scattering amplitudes obtained from the two methods of regularization are *inequivalent*. After renormalization, the calculation using a cutoff predicts a trivial scattering amplitude (in accord with Friedman's theorem) while the calculation using DR predicts a constant amplitude which in general is not constrained to vanish.

Of course, the five-dimensional delta function, while clearly illustrating possible disagreements between DR and cutoff schemes, is not otherwise intrinsically interesting. A problem of more relevance to physical calculations is studied in Sec. 3, where we consider potentials containing two derivatives of the delta function in $d = 3$, i.e. $V_3^{(2)}$ of Eq. (1). This potential has been used in effective field theory treatments of the nucleon-nucleon interaction [9]. In this problem DR and cutoff schemes predict different physical amplitudes. The on-shell T matrix obtained using DR is

$$T_{\mathrm{DR}}^{\mathrm{on}}(E) = -\frac{2\pi}{\mu} \frac{1}{1/(-a - r_e a^2 \mu E) - i\sqrt{2\mu E}}, \tag{3}$$

while that found using cutoff regularization is, in the infinite cutoff limit,

$$T_{\mathrm{cutoff}}^{\mathrm{on}}(E) = -\frac{2\pi}{\mu} \frac{1}{-1/a + r_e \mu E - i\sqrt{2\mu E}}; \quad r_e \leq 0, \tag{4}$$

where a and r_e are the scattering length and effective range. Observe that the scattering amplitudes obtained using the two schemes have different functional forms. Moreover, the two results do not apply under the same conditions. In DR it appears that the amplitude can always be renormalized regardless of the magnitude or sign of r_e and a. In contrast, in the cutoff regularization scheme the non-linear renormalization conditions can only be satisfied if the effective range is negative. In Sec. 4 we discuss a generalization of these results which is based on a result derived by Wigner [13, 14]. We show that, no matter what potential

$V_3^{(n)}$ is used, in the cutoff scheme the renormalized on-shell amplitude always obeys

$$\frac{\partial}{\partial E}\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}(E)}\right) \le 0. \tag{5}$$

No such constraint applies to the on-shell amplitude generated by DR.

The source of the inequivalence between these results obtained using different regularization methods is easily diagnosed. It is DR's prescription that all power-law divergences be discarded. By contrast, in cutoff regularization these divergences are retained and must be dealt with by renormalization. In perturbative calculations this difference in the treatment of power-law divergences is irrelevant after renormalization, since to a given order in perturbation theory such divergences may always be absorbed into counterterms [1]. The key point is that DR often involves more than *just* regularizing integrals. It also does a certain amount of renormalization by automatically subtracting all power-law divergences. There is therefore an implicit assumption that these effects can always be absorbed into counterterms. Our examples show that in certain non-perturbative calculations this partial renormalization that is automatically and implicitly done in DR can lead to unphysical results. This is because in non-perturbative problems, the renormalization conditions relating bare parameters to experimental data are generally non-linear, and it is not always possible to absorb all power-law divergences into counter terms. These issues are discussed and our conclusions drawn in Sec. 5.

2 The delta-function potential in d dimensions

In this section we discuss scattering from the delta-function potential in d dimensions. We solve the Lippmann-Schwinger equation for this potential in momentum space at energies $E \ge 0$, using both a sharp cutoff and DR, and demonstrate that the resultant renormalized amplitudes are different in any odd dimension $d > 3$. The generalization of our arguments to a smooth momentum-space cutoff is straightforward. Solutions for the delta-function potential where the regularization is done using position-space cutoffs may be found in Refs. [5, 6]. Discussions of the rigorous definition of such potentials via the construction of self-adjoint extensions of the Laplacian on the space of smooth functions compactly supported away from the origin appear in Refs. [15, 16, 17]. Here we avoid these mathematical questions and simply solve the Lippmann-Schwinger equation for these potentials using the regularization and renormalization procedure defined in the Introduction.

We wish to solve the Lippmann-Schwinger equation:

$$T(p',p;E) = V(p',p) + \int \frac{d^dk}{(2\pi)^d}\, V(p',k)\frac{1}{E^+ - \frac{k^2}{2\mu}}T(k,p;E), \tag{6}$$

where $E^+ \equiv E + i\epsilon$, with $E \ge 0$, and μ is the reduced mass in the two-body system. In momentum space, the delta-function potential is given by

$$V(p',p) = C. \tag{7}$$

[1] Sometimes symmetries require that the sum of contributions from power law divergences and their counterterms vanish. DR is particularly convenient for such cases. Consequently, it is well suited to both chiral theories and gauge theories.

From Eq. (6) it is apparent that the corresponding T-matrix has no momentum dependence, i.e.,

$$T(p', p; E) = T^{\rm on}(E). \tag{8}$$

It follows that

$$\frac{1}{T^{\rm on}(E)} = \frac{1}{C} - I(E), \tag{9}$$

where

$$\begin{aligned} I(E) &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{E^+ - \frac{k^2}{2\mu}}, & (10)\\ &= \frac{4\pi^{d/2}\mu}{\Gamma(d/2)(2\pi)^d} \int_0^\infty dk \frac{k^{d-1}}{2\mu E^+ - k^2}. & (11) \end{aligned}$$

We have done the angular integration, thereby obtaining a factor equal to the surface area of the unit sphere in d dimensions. Using the identity

$$\int_0^\infty dk\, f(k) \frac{1}{2\mu E^+ - k^2} = P \int_0^\infty dk\, f(k) \frac{1}{2\mu E - k^2} - i \frac{\pi}{2\sqrt{2\mu E}} f(\sqrt{2\mu E}), \tag{12}$$

where P denotes a principal value integral, we see that

$$I(E) = \frac{4\pi^{d/2}\mu}{\Gamma(d/2)(2\pi)^d} P \int_0^\infty dk\, \frac{k^{d-1}}{2\mu E - k^2} - i \frac{2\pi^{d/2+1}\mu}{\Gamma(d/2)(2\pi)^d} (2\mu E)^{d/2-1}. \tag{13}$$

The real part of the integral $I(E)$ is clearly divergent for all $d \geq 2$, and so requires some form of regularization. Suppose that we calculate this integral by isolating all divergent pieces and introducing a momentum cutoff β. In order to do this we make use of the relation:

$$P \int_0^\infty dk\, \frac{k^n}{2\mu E - k^2} = -\int_0^\infty dk\, k^{n-2} + 2\mu E\, P \int_0^\infty dk\, \frac{k^{n-2}}{2\mu E - k^2}, \tag{14}$$

which holds for all $n \geq 2$. This leads to the following series expressions:

$$P \int_0^\infty dk\, \frac{k^{d-1}}{2\mu E - k^2} = \begin{cases} -\frac{\beta^{d-2}}{d-2} - 2\mu E \frac{\beta^{d-4}}{d-4} - \ldots - (2\mu E)^{(d-3)/2}\beta & d \text{ odd} \\ -\frac{\beta^{d-2}}{d-2} - \ldots - (2\mu E)^{d/2-2}\frac{\beta^2}{2} + \frac{1}{2}(2\mu E)^{d/2-1} \log(\frac{2\mu E}{\beta^2 - 2\mu E}) & d \text{ even,} \end{cases} \tag{15}$$

provided that $\beta^2 > 2\mu E$. Note that in the case where d is odd we have not regulated the integral

$$P \int_0^\infty dk\, \frac{1}{2\mu E - k^2}, \tag{16}$$

since this convergent integral is equal to zero. If a smooth cutoff is used these results still hold, but different numerical coefficients reside in front of the terms in the series in Eq. (15). The argument which follows is completely unaffected by this change.

By contrast, the standard methods of evaluation used in dimensional regularization (see, e.g. [18]) lead to:

$$\mathrm{Re}[I(E)] = -\frac{2\mu}{(4\pi)^{d/2}}\,\mathrm{Re}[(-2\mu E)^{d/2-1}]\,\Gamma(1-d/2). \tag{17}$$

Note that for any $E > 0$ and any odd d this yields

$$\mathrm{Re}[I(E)] = 0. \tag{18}$$

In order to have either of the results (15) or (17) for $I(E)$ yield physical quantities we must renormalize the amplitude $T(p', p; E)$. Since for this potential $T(p', p; E) = T^{\mathrm{on}}(E)$ we choose some energy $E_0 \geq 0$ and write the renormalization condition as

$$T(p', p; E_0) = t, \tag{19}$$

where t is finite.

First, consider the case $d = 2$. In this case cutoff regularization gives:

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}(E)}\right) = \frac{1}{C_{\mathrm{cutoff}}} + \frac{\mu}{2\pi}[\log(\beta^2 - 2\mu E) - \log(2\mu E)]. \tag{20}$$

So choose

$$\frac{1}{C_{\mathrm{cutoff}}} = \mathrm{Re}\left(\frac{1}{t}\right) - \frac{\mu}{2\pi}[\log(\beta^2 - 2\mu E_0) - \log(2\mu E_0)]. \tag{21}$$

Then, in the limit as $\beta \to \infty$:

$$\frac{1}{T^{\mathrm{on}}(E)} = \mathrm{Re}(\frac{1}{t}) + \frac{\mu}{2\pi}\log(\frac{E_0}{E}) + i\frac{\mu}{2}. \tag{22}$$

This is to be compared to what occurs in DR. There

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}}\right) = \frac{1}{C_{\mathrm{DR}}} + \frac{\mu}{2\pi}\lim_{d\to 2}\Gamma(1-d/2)\mathrm{Re}[(-2\mu E)^{d/2-1}]. \tag{23}$$

Thus,

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}}\right) = \frac{1}{C_{\mathrm{DR}}} + \frac{\mu}{2\pi}\lim_{d\to 2}\left[\left(\frac{1}{1-d/2} - \gamma + \ldots\right)(1 + (d/2-1)\log(2\mu E))\right], \tag{24}$$

where γ is the Euler number. So, we choose:

$$\frac{1}{C_{\mathrm{DR}}} = \mathrm{Re}\left(\frac{1}{t}\right) - \frac{\mu}{2\pi}\left[\lim_{d\to 2}\frac{1}{1-d/2} - \log(2\mu E_0) - \gamma\right], \tag{25}$$

and recover Eq. (22).

Next, take the case $d = 3$. There cutoff regularization gives

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}(E)}\right) = \frac{1}{C_{\mathrm{cutoff}}} + \frac{\mu\beta}{\pi^2}, \tag{26}$$

while DR yields

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}(E)}\right)=\frac{1}{C_{\mathrm{DR}}}. \tag{27}$$

Thus, by choosing C appropriately, in both cases the result

$$\frac{1}{T^{\mathrm{on}}(E)}=\mathrm{Re}\left(\frac{1}{t}\right)+i\frac{\mu\sqrt{2\mu E}}{2\pi}, \tag{28}$$

derived by Weinberg [8] and Kaplan *et al.* [9] is obtained. Therefore, in $d=2$ and $d=3$, both regularization schemes lead to equivalent physical results.

Note that in $d=2$ and $d=3$ C_{cutoff} must be negative as the cutoff is taken to infinity. This corresponds to an attractive bare potential. Therefore, if the bare potential is repulsive, the scattering amplitude must be trivial, in accord with Friedman's theorem [11, 12]. In $d=2$, DR is consistent with Friedman's theorem if the coefficient of the $1/(d-2)$ pole is positive as $d\to 2$. In $d=3$, DR places no constraint on the bare potential. However, the seriousness of this inconsistency with Friedman's theorem is open to question, since the sign of the bare potential is not necessarily physically significant.

Next consider the calculation of $I(E)$ in any even dimension $d\geq 4$. If cutoff regularization is used then Eq. (15) indicates that the resulting $I(E)$ contains an energy-dependent divergence. Consequently, although the resulting amplitude may be made finite at energy E_0 by choosing $1/C_{\mathrm{cutoff}}$ appropriately, it is impossible to make $1/T^{\mathrm{on}}$ finite for any other energy. Therefore $1/T^{\mathrm{on}}$ diverges unless we require that C_{cutoff} vanishes, and we recover the result expected from Friedman's theorem; the scattering amplitude is trivial [11].

In DR in even dimensions, say $d=2w$,

$$\frac{1}{T^{\mathrm{on}}_{\mathrm{DR}}(E)}=\frac{1}{C_{\mathrm{DR}}}+\frac{2\mu}{(4\pi)^w}(-2\mu E)^{w-1}\Gamma(1-w)+i\frac{2\pi\mu}{\Gamma(w)(4\pi)^w}(2\mu E)^{w-1}. \tag{29}$$

Since $\Gamma(1-w)$ is divergent for all positive integers for all $w\geq 2$ an energy-dependent divergence appears here too. Therefore for even dimensions $d\geq 4$ both regularization schemes predict a trivial scattering amplitude:

$$T^{\mathrm{on}}(E)=0. \tag{30}$$

The case of odd dimensions $d\geq 5$ is more interesting. Once again, in cutoff regularization the integral $I(E)$ contains an energy-dependent divergence and so the renormalized amplitude is trivial, in accord with Friedman's theorem:

$$T^{\mathrm{on}}_{\mathrm{cutoff}}(E)=0. \tag{31}$$

By contrast, in DR the real part of the integral $I(E)$ is zero in each odd dimension $d\geq 3$. So,

$$\frac{1}{T^{\mathrm{on}}_{\mathrm{DR}}(E)}=\frac{1}{C_{\mathrm{DR}}}+i\frac{2\pi\mu}{\Gamma(d/2)(4\pi)^{d/2}}(2\mu E)^{d/2-1}. \tag{32}$$

Renormalization can now be performed, leading to a nontrivial scattering, by choosing $1/C_{\mathrm{DR}}=\mathrm{Re}(1/t)$. The cutoff and dimensionally regularized theories are therefore inequivalent. Indeed, the amplitude found using DR corresponds to one obtained from a Schrödinger

equation with a non-Hermitian pseudo-potential, as shown explicitly in five dimensions by Grossman and Wu [19].

However, the sense in which the two calculations starting with the potential $V^{(0)}$ are inequivalent is somewhat limited. After all, one could choose $C_{\rm DR} = 0$ in the dimensionally regularized theory and so obtain a trivial amplitude there as well. Nonetheless, there is an important way in which the two approaches are inequivalent—in one case one concludes that the theory is capable of yielding a finite, nonzero scattering amplitude and in the other case one concludes that it is not. In the next section we present a more compelling example of inequivalence, where the two regularization schemes give different, finite, nonzero scattering amplitudes.

3 A delta function and its derivative-squared in three dimensions

In this section we consider the example of the potential $V_3^{(2)}$, i.e. a potential in three dimensions which consists of a delta function plus two derivatives thereof. We show that for this potential cutoff regularization and DR give finite inequivalent results for the renormalized S-wave scattering amplitude.

In momentum space this potential may be written

$$V(p', p) = C + C_2(p^2 + p'^2). \tag{33}$$

We want to insert this into the Lippmann-Schwinger equation (6) and solve for $d = 3$. One way to do this is to observe that V may be written as a two-term separable potential

$$V(p', p) = \sum_{i,j=0}^{1} p'^{2i} \lambda_{ij} p^{2j}, \tag{34}$$

where the matrix λ is

$$\{\lambda_{ij}\}_{i,j=0}^{1} = \begin{pmatrix} C & C_2 \\ C_2 & 0 \end{pmatrix}. \tag{35}$$

The solution to the Lippmann-Schwinger equation then takes the form

$$T(p', p; E) = \sum_{i,j=0}^{1} p'^{2i} \tau_{ij}(E) p^{2j}, \tag{36}$$

where τ obeys the matrix equation:

$$\tau(E) = \lambda + \lambda \mathcal{I}(E) \tau(E), \tag{37}$$

with

$$\mathcal{I}(E) = \begin{pmatrix} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E^+ - \frac{k^2}{2\mu}} & \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E^+ - \frac{k^2}{2\mu}} \\ \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E^+ - \frac{k^2}{2\mu}} & \int \frac{d^3k}{(2\pi)^3} \frac{k^4}{E^+ - \frac{k^2}{2\mu}} \end{pmatrix}. \tag{38}$$

Now using the relation (14) we see that:

$$\mathcal{I}(E) = \begin{pmatrix} I(E) & I_3 + 2\mu E\, I(E) \\ I_3 + 2\mu E\, I(E) & I_5 + 2\mu E\, I_3 + 4\mu^2 E^2\, I(E), \end{pmatrix}, \tag{39}$$

with Eq. (13) for $d = 3$ defining $I(E)$ and

$$I_5 \equiv -2\mu \int \frac{d^3k}{(2\pi)^3} k^2; \quad I_3 \equiv -2\mu \int \frac{d^3k}{(2\pi)^3}. \tag{40}$$

We are now in a position to solve the algebraic equation (37) for $\tau(E)$. This gives

$$\tau(E) = \frac{1}{\det \tau^{-1}} \begin{pmatrix} -\frac{C}{C_2^2} - I_5 - 2\mu E I_3 - 4\mu^2 E^2 I(E) & I_3 + 2\mu E\, I(E) - \frac{1}{C_2} \\ I_3 + 2\mu E\, I(E) - \frac{1}{C_2} & -I(E) \end{pmatrix}, \tag{41}$$

with,

$$\det \tau^{-1} = \left[\frac{C}{C_2^2} I(E) + I_5 I(E) - 2\mu E\, I_3 I(E) - (\frac{1}{C_2} - I_3)^2 + 4\mu E I(E) \frac{1}{C_2} \right] \tag{42}$$

When this expression is inserted into Eq. (36), we find for the on-shell t-matrix:

$$\frac{1}{T^{\rm on}(E)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + 2\mu E C_2 (2 - C_2 I_3)} - I(E). \tag{43}$$

The integrals I_3, I_5 and $\mathrm{Re}(I(E))$ are all divergent, and so this amplitude requires regularization and renormalization. Indeed, *a priori* it is not clear that renormalization will be possible, since we have three distinct infinities and only two counterterms.

In fact this renormalization can be carried a certain distance without making reference to any particular regularization scheme. We choose as renormalization parameters the experimental values of the scattering length, a, and the effective range, r_e. In other words, we fix C and C_2 by demanding that

$$\frac{1}{T^{\rm on}(E)} = -\frac{\mu}{2\pi} \left(-\frac{1}{a} + r_e \mu E + O(\mu^2 E^2) - i\sqrt{2\mu E} \right). \tag{44}$$

Comparing this to Eq. (43) shows that the imaginary parts automatically agree (as is guaranteed by the unitarity of the Lippmann-Schwinger equation). Equating the real parts at $E = 0$ yields

$$\frac{\mu}{2\pi a} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5} - I(0). \tag{45}$$

Noting that,

$$\mathrm{Re}(I(E)) = I(0) \equiv I_1 = -2\mu \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2}, \tag{46}$$

this gives

$$\mathrm{Re}\left(\frac{1}{T^{\mathrm{on}}(E)}\right)=\frac{\mu/(2\pi a)-2\mu E\, I_1 A}{1+2\mu E\, A}, \tag{47}$$

with

$$A\equiv(\frac{\mu}{2\pi a}+I_1)\frac{C_2(2-C_2I_3)}{(C_2I_3-1)^2}. \tag{48}$$

From Eqs. (44), (47), and (48) the renormalization condition relating C_2 to the physical parameter r_e may be economically expressed as a condition on A:

$$A=\frac{\mu r_e}{4\pi}\left(I_1+\frac{\mu}{2\pi a}\right)^{-1}. \tag{49}$$

Let us consider the structure of this renormalization condition and the resulting amplitude (47) for both DR and cutoff regularization.

First, we regularize using DR. There

$$I_1=I_3=I_5=0. \tag{50}$$

Consequently, the renormalization condition (49) becomes

$$A=\frac{1}{2}ar_e, \tag{51}$$

with $A=C_2\mu/\pi a$. The term in the numerator of Eq. (47) disappears, leading to

$$\frac{1}{T^{\mathrm{on}}_{\mathrm{DR}}(E)}=-\frac{\mu}{2\pi}\left(\frac{1}{-a-a^2r_e\mu E}-i\sqrt{2\mu E}\right) \tag{52}$$

for all a and r_e, as found by Kaplan *et al.* [9].

By contrast, if a cutoff is used when the renormalization condition (49) is rewritten in terms of C_2 we have

$$\frac{\mu r_e}{4\pi}=\left(\frac{\mu}{2\pi a}+I_1\right)^2\left[\frac{1}{(C_2I_3-1)^2I_3}-\frac{1}{I_3}\right]. \tag{53}$$

In any scheme where a cutoff β is used to regulate the theory we have, on dimensional grounds:

$$\frac{I_1^2}{I_3}\sim\frac{1}{\beta}, \tag{54}$$

and thus the second term in Eq. (53) disappears in the limit that the cutoff is taken to infinity. Consequently, as $\beta\to\infty$

$$\frac{\mu r_e}{4\pi}\to\frac{1}{I_3}\left(\frac{I_1}{C_2I_3-1}\right)^2, \tag{55}$$

from which we see that $r_e\leq 0$, *independent of the value of* C_2, provided only that C_2 is real, i.e. the original bare Hamiltonian is Hermitian. It could be argued that the bare

Hamiltonian, being unobservable, is not required to be Hermitian. We will return to the issue of the physical significance of the bare Hamiltonian in Section 5 and in Ref. [10].

The important point is that, from Eqs. (53), (48) and (47) we see that the cutoff scheme yields an amplitude which, as the cutoff is taken to infinity, becomes

$$\frac{1}{T^{\rm on}_{\rm cutoff}(E)} = -\frac{\mu}{2\pi}\left(-\frac{1}{a} + r_e\mu E - i\sqrt{2\mu E}\right); \qquad r_e \leq 0. \tag{56}$$

Note that the type of cutoff regularization used here is irrelevant. In any scheme where the cutoff carries powers of momentum the behavior (54) occurs as the cutoff is removed, and we are led to the form (56) for the amplitude. This has a functionally different form to that obtained in Eq. (52) using DR. This means that even if $r_e \leq 0$ and the renormalization condition (49) can be satisfied, the DR results are still different from those based on a cutoff.

It is clear why DR leads to a different renormalized amplitude. The cause is the DR prescription of discarding all power-law divergences. The different results obtained for the form of the amplitude may be thought of as arising from the way the ratio A behaves in the two different regularization schemes. In DR $I_1 \equiv 0$, and $A = \frac{1}{2}ar_e$, so the term in the numerator of Eq. (47) disappears, thus leading to (52). However, in a cutoff scheme, as $\beta \to \infty$, $I_1 A \to \frac{\mu}{4\pi} r_e$, and so the term in the denominator of Eq. (47) disappears, leading to the form (56). Moreover, in the cutoff case as $\beta \to \infty$ A has a fixed sign, and so if $r_e > 0$ renormalization is simply impossible. Since the ratio I_1^2/I_3 has no meaning in DR the sign of A, and hence that of r_e, is unconstrained in that approach.

4 The Wigner bound on the amplitude generated by $V_3^{(2N)}$

We have seen that if cutoff regularization is used to give meaning to $V_3^{(2)}$ renormalization can only be performed if the effective range is negative. One might think that this constraint is a special feature of the calculation with $V_3^{(2)}$, but here we show that it is a result which holds no matter how many derivatives of the delta function are included in the potential (1). We do this by making a connection with the bound on r_e originally derived by Wigner [13], rederived by Fewster [17], and discussed in Ref. [14]. There it was shown that, for any energy-independent potential $V(r', r)$ which obeys

$$V(r', r) = 0 \text{ for all } r, r' > R, \tag{57}$$

the effective range is bounded by

$$r_e \leq 2\left(R - \frac{R^2}{a} + \frac{R^3}{3a^2}\right). \tag{58}$$

It was also shown that this bound applies even if the potential does not go strictly to zero, but merely decreases fast enough for the wave function to approach the asymptotic solution sufficiently quickly.

In order to make the bound of Ref. [14] apply to the problem we have been discussing here we need to show two things. First, we must show that the position-space arguments

of Ref. [14] can be translated into momentum space. Second, we must prove that, in the limit that the regulator is removed (i.e. the regulator mass goes to infinity), regulating the potential and then subsequently iterating it via the Lippmann-Schwinger equation is equivalent to formally iterating the interaction and then regulating all divergent integrals using a cutoff.

To explain the first point we consider the class of potentials

$$V_\beta(p',p) = g(\frac{p'^2}{\beta^2}) \left(\sum_{i,j=0}^{N} p'^{2i} \lambda_{ij} p^{2j} \right) g(\frac{p^2}{\beta^2}). \tag{59}$$

Here β is the cutoff parameter of the (sharp or smooth) cutoff function g. This function $g(x^2)$ obeys $g(0) = 1$ and $g(x^2) \to 0$ faster than $\frac{1}{x^{N+1/2}}$ as $x \to \infty$. (For instance, $g(x^2) = \exp(-x^2)$ is an acceptable choice.) In the limit $\beta \to \infty$ this class of potentials formally approaches the class of potentials we are considering in Eq. (1) with M_N of the λ_{ij}'s nonzero where

$$M_N = \begin{cases} \frac{N^2}{4} + N + \frac{3}{4} & \text{if } N \text{ is odd} \\ \frac{N^2}{4} + N + 1 & \text{if } N \text{ is even.} \end{cases} \tag{60}$$

Therefore once an amplitude is calculated using V_β M_N renormalization conditions are needed. Renormalization can be achieved by fixing β and then calculating the λ_{ij}s needed to reproduce the effective range expansion to order M_N for that value of β. Below we show that as $\beta \to \infty$ the amplitude obtained by this new procedure is identical to that derived via the method used in the previous section.

But first we show that the Wigner bound on the effective range (58) applies to the potential $V_\beta(p',p)$. Equation (58) was derived in position space. Taking the Fourier transform of (59) we get

$$V_\beta(r',r) = \tilde{g}(r'^2\beta^2) \left[\sum_{i,j=0}^{N} (-\overleftarrow{\nabla}'^{\,2})^i \lambda_{ij} (-\overrightarrow{\nabla}^{\,2})^j \right] \tilde{g}(r^2\beta^2), \tag{61}$$

where $\tilde{g}$ is the Fourier transform of $g(k^2)$ with respect to $\vec{k}$. For any g which, for arbitrary N, obeys the conditions given above no finite R exists for which $\tilde{g}$ goes strictly to zero for all $r, r' > R$. However, by choosing g appropriately we can, for β large enough, make $V_\beta(r,r')$ arbitrarily close to zero. Hence the wave function can be made to approach the asymptotic solution rapidly enough for the Wigner bound to apply, with the range $R \to 0$ as $\beta \to \infty$. Thus, as $\beta \to \infty$ Eq. (58) yields:

$$r_e \leq 0, \tag{62}$$

for the renormalized amplitude generated by V_β *regardless of the value of* N.

So, all that is left for us to prove is that iterating and then using cutoff regularization is equivalent to regulating via, e.g. Eq. (59), and then iterating. That is, we must show that, once the cutoff is taken to infinity, iterating the regularized potential V_β and renormalizing the coefficients is equivalent to formally iterating the potential $V_3^{(2N)}$, regulating all divergent integrals using a cutoff, and then renormalizing. To do this let us consider an unregulated

potential $V_3^{(2N)}$:

$$V(p', p) = \sum_{i,j=0}^{N} p'^{2i} \lambda_{ij} p^{2j}. \tag{63}$$

The t-matrix generated by this interaction has the form

$$T(p', p; E) = \sum_{i,j=0}^{N} p'^{2i} \tau_{ij}(E) p^{2j}, \tag{64}$$

where $\tau(E)$ is to be found using Eq. (37) with the matrix $\{\mathcal{I}\}_{i,j=0}^{N}$ defined by

$$\mathcal{I}_{ij}(E) = \int \frac{d^3k}{(2\pi)^3} \frac{k^{2(i+j)}}{E^+ - \frac{k^2}{2\mu}}. \tag{65}$$

Now using the recursion relation (14) we observe that

$$\mathcal{I}_{ij}(E) = \tilde{\mathcal{I}}_{ij}(E) + (2\mu E)^{i+j} I(E). \tag{66}$$

These arguments are similar to those used in solving the Lippmann-Schwinger equation for the potential V_β. However, one main difference is that there:

$$\mathcal{I}_{ij}(E) = \int \frac{d^3k}{(2\pi)^3} \frac{k^{2(i+j)}}{E^+ - \frac{k^2}{2\mu}} \left[g\left(\frac{k^2}{\beta^2}\right)\right]^2. \tag{67}$$

Nevertheless, since the integrals $\tilde{\mathcal{I}}_{ij}$ are all real and divergent, they are unaffected by the order in which regularization and iteration are performed.

Define

$$\tilde{\tau}^{-1}(E) = \lambda^{-1} - \tilde{\mathcal{I}}(E). \tag{68}$$

It follows that

$$\tau^{-1}(E) = \tilde{\tau}^{-1}(E) - \mathbf{g}(E) I(E), \tag{69}$$

where the matrix $\mathbf{g}(E)$ is defined by:

$$\{g\}_{i,j=0}^{N} = (2\mu E)^{i+j}. \tag{70}$$

If we define $\tilde{T}^{\rm on}$ via

$$\tilde{T}^{\rm on}(E) = \sum_{i,j=0}^{N} (2\mu E)^{i+j} \tilde{\tau}_{ij}(E) \tag{71}$$

then from Eq. (69) we obtain:

$$\frac{1}{T^{\rm on}(E)} = \frac{1}{\tilde{T}^{\rm on}(E)} - I(E). \tag{72}$$

Note that since the imaginary part of $I(E)$ is given by the formula (13) with $d = 3$, and the amplitude $\tilde{T}$ is real, this argument shows that the amplitude $T^{\rm on}$ will always be unitary.

Equation (68) implies that the amplitude $\tilde{T}^{\rm on}$ is unaffected by the order in which regularization and iteration are performed. Therefore, the only way the order of these two actions can affect $T^{\rm on}(E)$ is via the real part of the integral $I(E)$. We saw above that if regularization is performed after iteration $\mathrm{Re}(I(E)) = I_1$. By contrast, if the potential V_β is used, we have, for $2\mu E \ll \beta^2$,

$$\mathrm{Re}(I(E)) = I_1 - 2\mu\beta \sum_{i=1}^{\infty} c_i \left(\frac{2\mu E}{\beta^2}\right)^i, \tag{73}$$

where the numerical coefficients c_i are always of order one, with their exact value dependent upon the particular regulator used. Now let us consider what happens when renormalization is performed. Suppose that the renormalization conditions are chosen so as to enforce agreement with the effective range expansion for $\mathrm{Re}(\frac{1}{T^{\rm on}(E)})$ up to order M_N. Since $\mathrm{Re}(I(E))$ only appears in $\mathrm{Re}\left(\frac{1}{T^{\rm on}(E)}\right)$, M_N of the terms in the sum of Eq. (73) can be absorbed by adjusting the unrenormalized coefficients in V_β. The terms which are not absorbed via this step then appear in the renormalized amplitude at any finite β. However, in the limit when $\beta \to \infty$ they make no contribution to the final result. Indeed, once $\beta \to \infty$ none of the terms in (73) except I_1 play any role, either during or after renormalization. Thus the order of iteration and regularization is immaterial once the cutoff is taken to infinity.

Consequently the Wigner bound applies to V_β in the limit $\beta \to \infty$ and in that limit iterating V_β yields the same amplitude as iterating (63) and then using cutoff regularization. Hence, no matter which $V^{(2N)}$ is used, in the limit that the cutoff is removed from the theory the renormalized amplitude can only predict a negative effective range. In fact, in the limit that the range of the potential goes to zero Ref. [14] showed that

$$\frac{d}{d\bar{p}^2}(\bar{p}\cot\delta(\bar{p})) \leq 0, \tag{74}$$

for all $\bar{p} = \sqrt{2\mu E}$. But, everything said above for the effective range applies to higher derivatives of the real part of the inverse amplitude. Therefore, regardless of the potential $V^{(2N)}$ used, and regardless of the order in which cutoff regularization and iteration are done, the bound (74) applies to the resultant amplitude.

But what happens if DR is used instead of cutoff regularization? To answer this question consider Eq. (68). Since all the integrals $\tilde{\mathcal{I}}_{ij}$ are power-law divergent with no finite or logarithmically-divergent part, in DR,

$$\tilde{\mathcal{I}}_{ij}(E) = 0 \text{ for all } i, j = 1, 2, \ldots, N. \tag{75}$$

Therefore,

$$\tilde{\tau}_{DR}(E) = \lambda^{-1}. \tag{76}$$

As observed in the previous section $\mathrm{Re}(I(E)) = 0$ in DR also, thus

$$\frac{1}{T^{\rm on}_{DR}(E)} = \frac{1}{\tilde{T}^{\rm on}_{DR}(E)} + i\frac{\mu\sqrt{2\mu E}}{2\pi}, \tag{77}$$

with

$$\tilde{T}^{\text{on}}_{DR}(E) = \sum_{i,j=0}^{N} \lambda_{ij}(2\mu E)^{i+j}. \tag{78}$$

Therefore by appropriate choice of the M_N coefficients λ_{ij} one can fit the first M_N coefficients in the effective range expansion for $\frac{1}{T^{\text{on}}_{DR}(E)}$ with no constraints at all. So, the amplitude generated by DR is apparently not subject to the constraint (74).

An alternative mathematically elegant way to discuss the delta-function potential $V_3^{(0)}$ is in terms of the self-adjoint extension of the free Hamiltonian on a space of smooth functions compactly supported away from the origin [16]. It has been shown that obtaining $r_e > 0$ with a delta-function potential $V_3^{(n)}$ defined in ways analogous to this is impossible for a Hamiltonian which is a Hermitian operator on a space with positive-definite norm [17]. This is to be expected from the Wigner bound (58), which was originally derived assuming only unitarity and causality. Since using DR allows amplitudes with $r_e > 0$ we conclude that the results given by DR for delta-function potentials such as $V_3^{(2)}$ do not correspond to a conventional quantum mechanical calculation with truly zero-range potentials.

5 Discussion

We have found several quantum mechanical problems where DR and cutoff regularization schemes give different physical results. Our findings raise two important questions which we will discuss in this section. First, it is well known that for calculations in quantum field theory which are carried out to a given order in the (renormalized) coupling constant these two regularization schemes give equivalent renormalized amplitudes [2]. What then are the features of the non-perturbative calculations considered here which lead to results at variance with the familiar perturbative case? Second, since the potential (1) has no meaning before a regularization scheme is chosen, and yet different regularization schemes lead to different physical amplitudes, we ask in what sense the expression (1) is to be understood.

In order to address the first issue it is useful to give a simple illustrative example of perturbative regularization and renormalization to contrast with the non-perturbative calculations of this paper. Consider the perturbative treatment of the fermion self-energy in a theory of fermions interacting with scalars via a Yukawa coupling. The Lagrangian with physical masses and couplings takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2) + \bar{\psi}(i\,\partial\!\!\!/ - M)\psi - g\bar{\psi}\phi\psi. \tag{79}$$

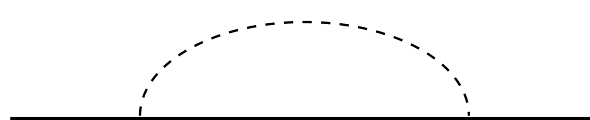

Figure 2: Fermion self-energy graph.

The graph shown in Fig. 2 corresponds to a fermion self-energy

$$\Sigma(p) = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{p} - \not{k} - M} \frac{i}{k^2 - m^2}. \tag{80}$$

This may be decomposed as $\Sigma(p) = A(p^2)\not{p} + B(p^2)$ with

$$\begin{aligned} A(p^2) &= \int d^4k\, f(p^2, k^2, p\cdot k) - \frac{1}{p^2} \int d^4k\, f(p^2, k^2, p\cdot k)\, p\cdot k, && (81) \\ B(p^2) &= M \int d^4k\, f(p^2, k^2, p\cdot k), && (82) \end{aligned}$$

where

$$f(p^2, k^2, p\cdot k) = \frac{g^2}{(2\pi)^4} \frac{1}{(p-k)^2 - M^2} \frac{1}{k^2 - m^2}. \tag{83}$$

Observe that both A and B are divergent, and, in particular, that the second integral in the expression for the coefficient function $A(p^2)$ contains a linear divergence.

In order to renormalize the free fermion Lagrangian we add the counterterms

$$(Z_2 - 1) i\bar{\psi}\not{\partial}\psi - \bar{\psi}(Z_2 M_0 - M)\psi. \tag{84}$$

The total fermion propagator, including $\Sigma(p)$, then becomes

$$d^{-1}(p) = Z_2(\not{p} - M_0) - A(p^2)\not{p} - B(p^2). \tag{85}$$

The counterterms Z_2 and M_0 are then chosen so that

$$\begin{aligned} M &= Z_2 M_0 + B(M^2), && (86) \\ 1 &= Z_2 - A(M^2), && (87) \end{aligned}$$

thus giving a propagator with unit residue at the pole $\not{p} = M$.

In DR the linear divergence in $A(M^2)$ is discarded, by prescription. However, Eq. (87) indicates that the neglect of this divergence merely results in an altered definition of the counterterm Z_2. Since the bare Lagrangian is unobservable the physical results obtained in DR and cutoff schemes will therefore be entirely equivalent. We have presented this example in some detail because it is a paradigm for the way perturbative renormalization is carried out in quantum field theory. The key feature here is that the renormalization is additive: the counterterms are linearly related to the divergences, and only the sum of the two has physical meaning. When this is the case a calculation done using cutoff regularization may be mapped to one using DR by simply changing the definition of all the counterterms. Under such a mapping the results for renormalized amplitudes will remain the same. However, this means that DR *is not solely* a regularization scheme. DR regulates divergent integrals, but also performs an implicit renormalization by setting all power-law divergences to zero. This automatic "subtraction" can be an asset in perturbation theory because it provides an economical way to ensure that symmetries like gauge invariance and chirality are manifest throughout a loop calculation.

In contrast to the above example regularization and renormalization in non-perturbative computations often have subtleties not present in the perturbative case. These subtleties occur because the relation between counterterms is more complicated than in perturbative quantum field theory. The renormalization conditions often take on a form different to that familiar from perturbative calculations, where a physical observable is usually expressed as the sum of a divergence and a counterterm.

Consider the case of the potential $V^{(0)}$ discussed in Section 2. After iteration there is one divergence and one counterterm $(1/C)$, with the two linearly related. In $d = 2$ and $d = 3$ the divergence can be absorbed into the counterterm. It is therefore no surprise that DR and cutoff schemes give equivalent results in these cases. However, in $d \geq 4$ energy-dependent divergences appear in the integral. These cannot be absorbed into the counterterm. In DR there are no divergences in odd dimensions $d \geq 3$ and so DR and cutoff schemes are inequivalent in these dimensions.

The potential of Section 3 provides a good example of the fundamental differences between perturbative and non-perturbative renormalization. Three divergences, I_1, I_3, and I_5 appear in the unrenormalized amplitude, and only two counterterms C and C_2 are present. The interrelation of these divergences and counterterms is highly nonlinear—a feature particular to non-perturbative renormalization. Under such circumstances DR is not necessarily equivalent to cutoff schemes. DR's disregard of linear divergences leads to a renormalized amplitude completely different to that found in cutoff regularization. Furthermore, the relative size of various divergences is meaningless in DR, and so the Wigner bound $r_e \leq 0$—which in cutoff schemes arises as a consequence of the non-linear relation of the various divergences—does not apply.

We turn now to our second issue: the interpretation of the potential (1). The potential defined by Eq. (1) is intrinsically meaningless: it makes sense only after regularization. Sections 2 to 4 demonstrate that in certain circumstances the physical amplitude which is obtained will be different in different regularization schemes. Therefore, we must choose which regularization scheme to use in order to give meaning to this otherwise nonsensical potential. Such a choice can only be made by considering the context in which such a delta-function potential might be used. We know of only one circumstance in which a delta-function potential can be profitably used. For this circumstance we will show that a cutoff scheme is the appropriate form of regularization.

Consider a general non-local potential $V(r, r')$ which is of range R. Suppose that the structure of V is tuned so that the first M_N terms (see Eq. (60)) in the effective range expansion are governed by a length scale $r_0 \gg R$. Despite this tuning, the $N + 1$th and higher terms in the effective range expansion will, in general, be governed by the scale R, not the scale r_0. If we attempt to model this potential by $V^{(N)}$ from Eq. (1) then we know, from our previous arguments, that DR and cutoff schemes give different results. There are two arguments in favor of the use of a cutoff scheme with the cutoff taken to infinity in this particular problem.

First, we know from Wigner's bound (58) that for $r_0 \gg R$ all coefficients in the effective range expansion beyond the scattering length must be negative. As was shown in Sections 3 and 4 cutoff regularization (with $\beta \to \infty$) reproduces this known feature of scattering from V. By contrast, DR imposes no such restriction on the effective range, shape parameter, etc.

Second, Eqs. (77) and (78) show that DR predicts that coefficients in the effective range expansion beyond those fit by the renormalization conditions will be of scale r_0. By contrast, in cutoff regularization one expects these higher-order effective range expansion coefficients to be of scale $1/\beta$, where β is the cutoff. If we interpret $\beta \sim 1/R$ this accords with our physical intuition about the size of these coefficients. Thus, cutoff regularization provides a physically intuitive procedure for giving meaning to the potential (1). It just reintroduces the range of the potential V we are trying to model via a delta function. Consequently, it will only be useful to take the limit $\beta \to \infty$ is there is truly a wide separation of scales in the problem, i.e. $R \ll r_0$. This condition is not obeyed in nucleon-nucleon scattering, except if only the scattering length is fitted in the ${}^1S_0$ channel. Therefore, our arguments here do not have direct relevance for effective field theory treatments of nucleon-nucleon scattering. The implications of this work for that problem will be elucidated in a forthcoming paper [10].

This demonstrates that for this class of problem cutoff regularization provides a controlled way of reintroducing the breakdown of delta-function behavior which we know must occur at short distances. On the other hand DR is an *ad hoc* modification of short-distance behavior. We would argue that both in perturbation theory and in a non-perturbative context one must verify that DR is a sensible procedure by regularizing using a cutoff scheme and comparing the calculated physical observables obtained via the two methods. Here we have exhibited an instance in which cutoff regularization gives results in accord with the behavior in an underlying theory while DR does not. At least in this problem, DR treats short-distance physics incorrectly.

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